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Contact geometry in Lagrangian mechanics

Pavol Ševera

Department of Theoretical Physics, Charles University, V Holešovičkách 2, 18000 Prague, Czech Republic

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Abstract

We present a picture of Lagrangian mechanics, free of some unnatural features (such as complete divergences). As a byproduct, a completely natural $U(1)$-bundle over the phase space appears. The correspondence between classical and quantum mechanics is very clear, e.g. no topological ambiguities remain. Contact geometry is the basic tool. © 1999 Elsevier Science B.V. All rights reserved.

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1. Introduction

In this paper we show how to get rid of some unnatural features of Lagrangian mechanics, such as multivaluedness and neglecting total divergences. There is almost nothing new (cf. [3]): we simply consider Hamilton–Jacobi equation and its characteristics. The only point is in introducing, instead of $M \times \mathbb{R}$, a principal G-bundle U over the space–time M, where $G = \mathbb{R}$ or $U(1)$. Even if U is trivial, only its fibred structure is to be considered as natural. Non-uniqueness of splitting the bundle into a product corresponds to "up to a total divergence" phrases. The Hamilton–Jacobi equation is simply a G-invariant hypersurface is the space of contact elements of U.

In quantization, the wave functions are sections of a line bundle associated to U, no topological ambiguity remains, so the correspondence classical $\leftrightarrow$ quantum is very clear. The space of characteristics $\mathfrak{Ch}$ carries a natural contact structure; the phase space $\mathfrak{Ph}$ emerges as the quotient $\mathfrak{Ch}/G$. Thus $\mathfrak{Ch} \to \mathfrak{Ph}$ is a principal $U(1)$-(or $\mathbb{R}$) bundle; the contact structure gives us a connection.

The plan of the paper is as follows: In Section 2 we review basic facts of contact geometry, its connection with symplectic geometry and geometrical quantization, with

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first-order PDE and the method of characteristics and with asymptotics of linear PDE. In Section 3 we introduce the point of view described above and discuss its correspondence with Lagrangians. For example, it may contain some additional topological information (obviously the topological quantization ambiguity has to be hidden somewhere). The bundle $\mathfrak{Ch} \to \mathfrak{Ph}$ and quantization are discussed in Section 4. We conclude with the fact that one can replace the group $U(1)$ by any Lie group almost without changing anything. Finally we mention the obvious open problem – what happens if we do not consider extremal curves, but surfaces etc.

2. Basic notions of contact geometry

A *contact structure* on a manifold M is a field of hyperplanes $HM \subset TM$ (a subbundle of codimension 1) satisfying a maximal non-integrability condition. It can be formulated as follows: similarly to any subbundle of TM, we have a map $\sigma : \wedge^2 HM \to TM/HM$ satisfying (and defined by) the fact that for any 1-form α on M, annulated on HM, the formula

$$\alpha(\sigma(u, v)) = \mathrm{d}\alpha(u, v)$$

holds for any $u, v \in H_x M, x \in M$. Alternatively, we may extend u and v to sections of HM; their commutator at x (when considered mod HM) is $\sigma(u, v)$. The maximal non-integrability condition requires σ to be a non-singular bilinear form everywhere. In that case, M is clearly odd-dimensional. Any two contact manifolds with the same dimension are locally isomorphic (a form of Darboux theorem).

We call a vector field on M *contact*, if its flow preserves the contact structure. There is a 1–1 correspondence between contact vector fields and sections of the line bundle TM/HM. More precisely, for any $w \in C^\infty(TM/HM)$ there is a unique contact v that becomes w when considered mod HM. The proof is easy: choose any v' that is w mod HM. As a rule, v' is not contact, so it generates an infinitesimal deformation of the contact structure – say $\beta : HM \to TM/HM$. But due to the non-degeneracy of σ there is a unique $v'' \in C^\infty(HM)$ producing the same deformation. Thus $v = v' - v''$ is the required contact field. The field w is called the *contact Hamiltonian* if v.

An important example of contact geometry emerges when M is a principal G-bundle over a symplectic manifold (N, ω), where $G = \mathbb{R}$ or $U(1)$. Suppose we are given a connection on M such that its curvature is ω. The horizontal distribution makes M into a contact manifold. We can use the connection 1-form α to identify sections of TM/HM (contact Hamiltonians) with functions on M. The local flow generated by a contact field v preserves the structure of G-bundle iff v is G-invariant, i.e. iff its contact Hamiltonian f is (the pullback of) a function on N. Then the field v is projected onto a well-defined vector v_N on N whose flow preserves ω; in fact, f is a Hamiltonian generating v_N. We may put these facts together: The Lie algebra $C^\infty(N)$ (with the Poisson bracket) is isomorphic to the Lie algebra of G-invariant contact fields on M. A function f on N and the corresponding Hamiltonian vector field v_N are combined together (f as the vertical part and v_N as the horizontal part) to form a contacts field v on M.

This point of view is useful in geometrical quantization. Here one considers a line bundle $L \to N$ associated to $M \to N$, and represents the Lie algebra $(C^\infty(N), \{\ ,\ \})$ by operators on the space $C^\infty(L)$. The sections of L are simply functions on M equivariant with respect to G and the action of a function $f \in C^\infty(N)$ on such a section is given by the derivative with respect to the corresponding contact vector field.

The classical example of a contact manifold is the space of contact elements (i.e. hyperplanes in the tangent space) of a manifold M, which we denote as CM. In other words, CM is the projective bundle associated with T^*M. The distribution $H(CM)$ is given as follows: take an $x \in CM$; it corresponds to a hyperplane H is $T_{\pi(x)}M$, where $\pi : CM \to M$ is the natural projection. Then $H_x(CM)$ is $(d_x\pi)^{-1}(H)$.

Contact geometry, in particular on CM, was invented to give a geometrical meaning to first-order partial differential equations and to Lagrange method of characteristics. Suppose $E \subset CM$ is a hypersurface; it will represent the equation. Any hypersurface $\Sigma \subset M$ can be lifted to CM: for any point $x \in \Sigma$ take the hyperplane $T_x\Sigma$ to be a point of the lift $\tilde{\Sigma}$. $\tilde{\Sigma}$ is a Legendre submanifold of CM, i.e. $T\tilde{\Sigma} \subset H(CM)$ and $\tilde{\Sigma}$ has the maximal dimension ($\dim CM = 2\dim\tilde{\Sigma} + 1$). Σ is said to solve the equation if $\tilde{\Sigma} \subset E$. This has a nice interpretation due to Monge: For any $x \in M$ we take the enveloping cone of the hyperplanes $\pi^{-1}(x) \cup E$ in T_xM. In this way we obtain a field of cones in M. Then Σ solves the equation if it is tangent to the cones everywhere.

Lie's point of view is to forget about M and to take as a solution any Legendre submanifold contained in E. Such a solution may look singular in M (singularities emerge upon the projection $\pi : CM \to M$). This definition uses only the contact structure on CM and thus allows using the entire (pseudo)group of contact transformations.

Now we will describe the method of characteristics. The hyperplane field $H(CM)$ cuts a hyperplane field HE on E (there may be points where the contact hyperplane touches E. Generally they are isolated and we will ignore them). The field HE does not make E into a contact manifold: the form σ becomes degenerate when we restrict ourselves from $H_x(CM)$ to H_xE. Thus at any $x \in E$ there appears a direction along which σ is degenerate. The integral curves of this direction field are called *characteristics*. For example, if the Monge cones coming from E are the null cones of some pseudo-Riemannian metrics on M then the projections of the characteristics are the light-like geodesics in M.

Generally, if F is a manifold with a hyperplane field HF, and the form $\sigma : \wedge^2 HF \to TF/HF$ has constant rank, then the bundle of kernels of σ, $KF \subset HF$, is integrable. Moreover, if one takes an open $U \subset F$ small enough, so that the integral manifolds of KF in U form a manifold $\mathfrak{Ch}$, then there is a well-defined contact structure on $\mathfrak{Ch}$ coming from the projection of HF. Coming back to the case of $E \subset CM$, it gives us a method of finding the Legendre submanifolds contained in E. Just take a submanifold $N \subset E$ such that $TN \subset H(CM)$ and $\dim CM = 2\dim N + 2$. Suppose that the characteristics intersect N transversally. Then their union forms a Legendre submanifold.

Let us look at vector fields on E with flow preserving the field HE; we shall call them contact, too. First of all, there are *characteristic vector fields*, i.e. fields touching the characteristics. Thus it is no longer true that if we choose a $w \in C^\infty(TE/HE)$ then there is a unique $v \in C^\infty(TE)$ equal to w mod HE: we can always add a characteristic field to v.

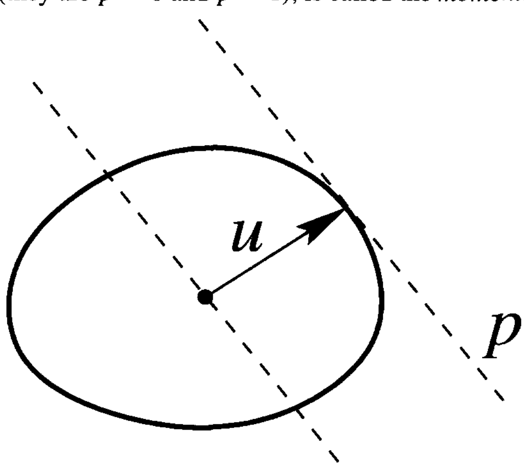
On the other hand, w cannot be arbitrary. The flow of a contact field has to preserve the characteristic foliation. If $\mathfrak{Ch}$ is the space of characteristics, each contact field on E can be projected onto a contact field on $\mathfrak{Ch}$ (recall $\mathfrak{Ch}$ is a contact manifold). This is the basis for conservation laws. For example if a contact field $v \in HE$ (i.e. $w = 0$) at a point $x \in E$ then $v \in HE(w = 0)$ along the characteristics γ_x running through x. Let us also notice that any contact vector field on E can be prolongated to a contact vector field on CM.

Hypersurfaces $E \subset CM$ often come from an equation of the type $Df = 0$, where $D : C^\infty(M) \to C^\infty(M)$ is a linear differential operator. Take the sybmol s_D of D (a function on T^*M defined by $(\mathrm{i}\lambda)^n s_D(\mathrm{d}g) = D\exp(\mathrm{i}\lambda g) + \mathrm{O}(\lambda^{n-1})$, $\lambda \to \infty$, where n is the degree of D and $g \in C^\infty(M)$). The equation $s_D = 0$ specifies a hypersurface $E \subset CM$. The singularities of solutions of $Df = 0$ are located on hypersurfaces solving the equation corresponding to E; also, if $f = a(x)\exp(\mathrm{i}\lambda S(x))$, $\lambda \to \infty$ is an asymptotic solution of $Df = 0$ then the levels $S(x) = \text{const.}$ solve the E- equation.

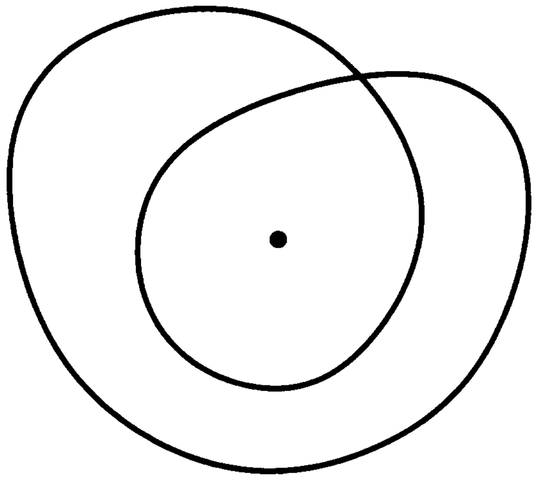
3. The geometry of Lagrangian mechanics

We shall deal with first-order variational principles. Suppose that at each point x of a manifold M (the space–time or an extended configuration space) there is a 1-homogeneous function $\Lambda_x : T_xM \to \mathbb{R}$ (and suppose everything is smooth outside the zero section of TM). Then on each oriented curve γ, Λ specifies a 1-form, so we may compute its integral $S(\gamma) = \int_\gamma \Lambda$ [1]. We are looking for extremals of S (in this paper extremal means stationary, i.e. with vanishing first variation; the actual local extremality is not assumed).

There are several reasons why this point of view (Finsler geometry) is not entirely satisfactory. First of all, even in the simplest problems, Λ_x is not defined on all T_xM, but only on an open conic subset. Even worse, Λ may be multivalued. An example is drawn on the following two figures. On the first one, we suppose that Λ_x is positive (outside 0). The figure represents the endpoints of vectors satisfying $\Lambda_x(v) = 1$; it is called the *wave diagram* in the beautiful elementary book [2]. The dashed lines represents a covector p corresponding to the drawn vector (they are $p = 0$ and $p = 1$); is called the *momentum*.



Obviously, we may use the field of wave diagrams instead of Λ. But we may work as well with diagrams of the following shape; they correspond to multivalued Λ's:



However, the real problem is that Λ is unnatural. The reason is that it is defined only up to a closed 1-form. For example, in the presence of an "electromagnetic field" $F \in C^\infty(\wedge^2 T^*M)$, $\mathrm{d}F = 0$, we take as the actual Λ (the one from which we compute S) $\Lambda + A$, where $\mathrm{d}A = F$. Of course A need not exist globally and it is not defined uniquely.

This problem appears also in Noether theorem: we take as an infinitesimal symmetry and vector field v whose flow preserves Λ up to some $\mathrm{d}f$. It is desirable to have a picture in which v is an actual symmetry.

A way out is in the following construction: Let $U \to M$ be a principal G-bundle, where $G = U(1)$ or $\mathbb{R}$ (you may imagine that we added the action S to M as a new coordinate; of course this interpretation is rather limited). Suppose we are given a G-invariant hypersurface $E \subset CU$; we are interested in its characteristics. Their projections to M are the extremals for certain (multivalued) Λ (if $c_1(U) \neq 0$ then either Λ exists only locally or we must admit an elmg. field F). We simply replaced Λ by the corresponding Hamilton–Jacobi equation E, but the new point of view is rid of the problems listed above. *For this reason we take $E \subset CU$ and its characteristics as fundamental and the Lagrangian Λ as a derived and sometimes ill-defined notion.*

The correspondence between E and Λ is as follows: Let α be an arbitrary connection 1-form on U. To find the wave diagram at a point $x \in M$, take a point $y \in U$ above x. The intersection of the Monge cone in T_yU with the hyperplane $\alpha = 1$ is the wave diagram. We have to take the curvature F as the elmg. field. We see that the transformation $\Lambda \to \Lambda + A$, $F \to F - \mathrm{d}A$ (A a 1-form) corresponds simply to a change of the connection.

If we start with Λ and F, we have to suppose that the periods of F are integral (or at least commesurable) to find a U admitting a connection with F as the curvature. Notice that if $H^1(M, G) \neq 0$, the picture $E \subset CU$ contains more information than the pair (Λ, F).

The inequivalent choices of U together with a connection correspond to the elements of the group $H^1(M, G)$ (this group acts there freely and transitively). The subgroup $H^1(M, \mathbb{Z})\otimes G$ corresponds to equivalent U's (with inequivalent connections); if $G = U(1)$, even the quotient group may be non-trivial (it is $\operatorname{Tor} H^2(M, \mathbb{Z})$).[1] These ambiguities are clearly connected with quantization.

A well-known example is the following: Let the Monge cones on U be the light cones of a Lorentzian metric and suppose the vector field u_G generating the action of G is space-like. As a connection on U take the orthogonal complements of u_G. Then the wave diagrams are the (pseudo)spheres of a Lorentzian metric on M. This picture describes a charged relativistic particle and its antiparticle in an elmg. field given by the curvature of the connection.[2] In the non-relativistic limit the field u_G becomes light-like and the antiparticle disappears.

Let us look at Noether theorem. In the (A, F)-picture one takes as a symmetry a vector field v together with a function f satisfying

$$v(A) + F(v, .) + \mathrm{d}f = 0$$

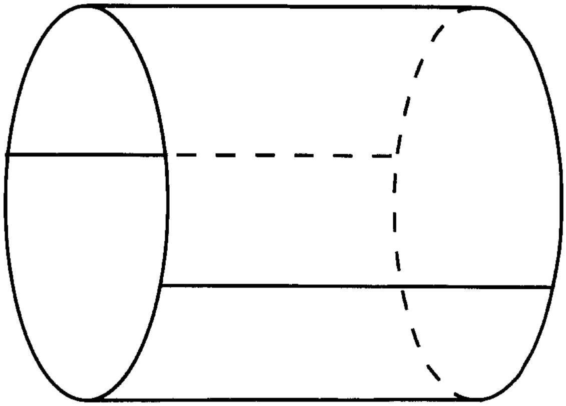
($v(\cdot)$ denotes the Lie derivative); then $p(v) + f$ is constant on extremals. But for $E \subset CU$ we simply take a G-invariant vector field on U preserving E. In fact one easily sees the full statement of Noether theorem [5], claiming a 1–1 correspondence between conservation laws and G-invariant contact fields on E modulo characteristic fields.

4. A $U(1)$-bundle over the phase space and quantization

Let us suppose that the characteristics form a manifold $\mathfrak{Ch}$ (i.e. we suppose that the characteristic foliation of E is sectionable and (if necessary) that $\mathfrak{Ch}$ is Hausdorff). $\mathfrak{Ch}$ inherits a contact structure. Notice that E is a G-bundle; if we suppose that the vector field generating the action of G is nowhere tangent to the characteristics then $\mathfrak{Ch}$ becomes a G'-bundle where $G' = G/H$ and $H \subset G$ is discrete (here we require E to be connected). Its base $\mathfrak{Ph} = \mathfrak{Ch}/G'$ is the phase space. Where the contact hyperplanes on $\mathfrak{Ch}$ may be used as a connection for $\mathfrak{Ch} \to \mathfrak{Ph}$, the curvature is the usual symplectic form on $\mathfrak{Ph}$. The points of $\mathfrak{Ph}$ where this is impossible (i.e. where the fibres are tangent to the contact hyperplanes) are not included in the usual phase space and they should be regarded as ideal. For example, the full $\mathfrak{Ph}$ of a relativistic particle in $1 + 1$-dimensions is on the following picture:

[1] We should use the additive group $\mathbb{R}/\mathbb{Z}$ instead of the multiplicative $U(1)$ to make the notation $H^1(M, \mathbb{Z})\otimes G$ coherent, but I hope the reader will not mind it.

[2] The connection dissects each light cone in U into two halfs. Thus the light-like geodesics in U (the characteristics) are (at least locally, and globally if there is a time orientation) divided into three classes; two of them are projected onto particles and antiparticles worldlines, respectively, while the curves in the third class are horizontal and they are projected onto light-like geodesics in M.



One half of the cylinder corresponds to particles, the other half to antiparticles and the connecting lines (the locus of the "ideal points") to light-like geodesics.

We see that there is a completely natural $U(1)$- or $\mathbb{R}$-bundle $\mathfrak{Ch}$ over the phase space, together with a natural connection. It is important in view of the use of such a bundle in quantization. Notice that $\mathfrak{Ch}$ is even prior to $\mathfrak{Ph}$.

Let us now look at quantization using wave functions in M. This may have nothing to do with quantum mechanics: we simply look for a wave equation that leads to given classical picture in a limit. Usually, one considers linear equations $D_h f = 0$ (h being a parameter in D) and looks for the high-frequency asymptotics as $h \to 0$ and the wavelength is of order h (see e.g. [4]). It is however much nicer if D is fixed; an outline of the theory was given at the end of Section 2. Thus let D be a G-invariant linear diff. operator on U. If we consider only G-equivariant functions (with the weight $1/h$), we get an operator D_h on the corresponding associated bundle.

For example, Schroedinger equation comes from

$$\left(\frac{1}{2m}\Delta + V(x,t)\frac{\partial^2}{\partial s^2} + \frac{\partial^2}{\partial s \partial t}\right)\psi(x,t,s) = 0,$$

where s is the new coordinate (here $U = M \times \mathbb{R}$): just notice that $\partial/\partial s$ becomes $\mathrm{i}/\hbar$ for ψ with the weight $1/\hbar$.

Let $E \subset CU$ be given by $s_D = 0$ where s_D is the symbol of D (notice that the Monge cone in T_xU is dual to the cone $s_{D,x} = 0$ in T_x^*U). In the obvious sense the equation $D_h f_h = 0$ gives the classical E-system as $h \to 0$. For example, take a (non-equivariant!) solution of $Df = 0$ with a singularity on a narrow strip along a characteristic of E. If we take the Fourier component f_h for $h \to 0$, it is significantly non-zero only close to the projection of the characteristic to M. Perhaps an interesting point is that the equation $Df = 0$ contains $D_h f_h = 0$ for any h.

Thus given E, quantization simply means a G-invariant D giving E by $s_D = 0$. Of course, the Monge cones of E have to be algebraic.

Finally, let us return to $\mathfrak{Ch} \to \mathfrak{Ph}$. We have a situation typical to integral geometry: $\mathfrak{Ch} \leftarrow E \to U$. In geometrical quantization one considers sections of bundles associated to $\mathfrak{Ch} \to \mathfrak{Ph}$, but here we take all possible h's at once, so we consider all the functions on $\mathfrak{Ch}$ instead. One should expect a correspondence between certain such functions and functions on U satisfying $Df = 0$. A polarization on $\mathfrak{Ph}$ gives us a G-invariant Legendrian foliation (if it is real) or (if it is completely complex) a G-invariant (codimension 1 and non-degenerate) CR-structure on $\mathfrak{Ch}$. The foliation gives us a complete system of solution of the Jacobi–Hamilton equation. Thus functions on $\mathfrak{Ch}$, constant on the leaves of the foliations, correspond to solutions of $Df = 0$ that are (integral) linear combinations of functions singular along hypersurfaces in the complete system. The CR-case is somewhat more complicated.

The discussion above is useless in this complete generality (and several important points were ommited), but it might be interesting for some classes of D's.

5. Conclusion

In the present paper G was always one-dimensional, but one can consider a principal G-bundle $U \to M$ and a hypersurface $E \subset CU$ for another Lie group G. The manifold $\mathfrak{Ch}$ is still contact, but $\mathfrak{Ph} = \mathfrak{Ch}/G$ is no longer symplectic; it carries only an analogue of symplectic structure. Characteristics of E represent particles in a Yang–Mills field. We can also consider a G-invariant operator $D : C^\infty(U) \to C^\infty(U)$. Suppose V is a G-module and the dual V^* contains a cyclic vector α. Let $\mathfrak{I}$ be the ideal in $\mathfrak{U}(\mathfrak{g})$ of the elements annulating α. Then we can embed V into the regular representation (namely onto the functions annulated by $\mathfrak{I}$) via $v \to \alpha(g \cdot v)$. In this way the functions on U annulated by $\mathfrak{I}$ are sections of the vector bundle associated to V. Thus D becomes an operator on these sections. We see the situation is quite analogous to one-dimensional G.

Perhaps the real problem is to go from extremal curves to surfaces and higher. The problems with Lagrangians remain the same.

Acknowledgements

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